Section 4: Applications of Exponential Functions (Text book – Section 6.3 and 6.4)

“The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth.”


When diseases grow exponentially (H1N1, Aids (when it first came on the scene in the 80s...but in Africa today is still growing exponentially))...the World Health Organization will often intervene....i.e....SARs virus 10 years ago in Toronto, Bird Flu, West Nile Virus...etc

Applications types we will deal with are:

1) Growth: Investments (money), bacterial growth, populations, Doubling time

2) Decay: vehicle depreciation, half-life (carbon-dating),
Section 4: Applications of Exponential Functions (Textbook – Section 6.3 and 6.4)

**How do we know if a set of data can be modeled using an exponential function?**

Whenever a quantity changes by the same factor (gets multiplied or divided by the same value) each time, then it can be modeled by an exponential function.

**Examples:**
- a population doubles each year.
- the amount of money in a bank account increases by 0.1% each month.
- the mass of a radioactive substance decreases by $\frac{1}{2}$ every 462 years

Many real world phenomena can be modeled by functions that describe how things grow or decay as time passes.
In the exponential equation \( y = a(b)^x \)

- \( a \) is the initial amount (ie. the \( y \)-intercept)
- \( b \) is the common ratio \( \left(\frac{y_2}{y_1}\right) \)
  - if \( b > 0 \), it is an increasing function (growth)
  - if \( 0 < b < 1 \), it is a decreasing function (decay)

Ex. Write the exponential equation that represents each table of values.

a) 

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>50</td>
<td>87.5</td>
<td>153.13</td>
<td>267.97</td>
<td>468.95</td>
</tr>
</tbody>
</table>

\[
a = 50 \quad b = 1.75 \quad \text{growth or decay: growth}
\]

\[
equation: \quad y = 50(1.75)^x
\]

b) 

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>23000</td>
<td>19550</td>
<td>16618</td>
<td>14125</td>
<td>12006</td>
</tr>
</tbody>
</table>

\[
a = 23000 \quad b = 0.85 \quad \text{growth or decay: decay}
\]

\[
equation: \quad y = 23000(0.85)^x
\]
**Half-Life Problems ( Decay)**

In a half life problem, the amount of a substance decreases by 1/2 every fixed number of years. The formula is:

\[ A(t) = A_0 \left( \frac{1}{2} \right)^{t/h} \]

where...
- \( A_0 \) represents the initial amount of the substance that was present.
- \( t \) represents the time
- \( h \) represents the half-life (amount of time taken for the substance to decrease by 1/2). It corresponds to that point on the graph of the function where the value of the function is half of its initial value \( A_0 \).
- \( A(t) \) is the amount of the substance present at time \( t \).

**NOTE:** Some other applications might have an equation given with a number other than 1/2 as the base in the power. For example, if a quantity doubles, the base will be 2, if it triples, the base will be 3, etc.
Examples:

1. A radioactive isotope decays at a rate described by the function
   \[ A(t) = 600 \left( \frac{1}{2} \right)^{\frac{t}{5000}} \], where \( t \) is the time in years and \( A(t) \) is the amount of isotope remaining in grams.

A) Determine the initial mass of the isotope.

\[ 600 \text{ g} \]

B) How long will it take to be reduced to half of its original amount?

1500 yr.

C) What is the mass of the isotope after 3000 years?

\[ t = 3000 \]

\[ A(t) = 600 \left( \frac{1}{2} \right)^{\frac{3000}{5000}} \]

\[ A(3000) = 600 \left( \frac{1}{2} \right)^{\frac{3}{5}} \]

\[ A(3000) = 600 \left( \frac{1}{4} \right) \]

\[ A(3000) = 150 \text{ g} \]

So after 3000 years there is 150 g left.

D) How long will it take to be reduced to one-eighth of its original amount?

\[ \frac{1}{8} \times 600 \quad \times \quad \text{find } t \quad \text{when } \quad A(t) = 75. \]

\[ = 75 \]

\[ = A(t) \]

\[ A(t) = 600 \left( \frac{1}{2} \right)^{\frac{t}{5000}} \]

\[ 75 = 600 \left( \frac{1}{2} \right)^{\frac{t}{5000}} \]

\[ \frac{75}{600} = \frac{1}{2}^{\frac{t}{5000}} \]

\[ \frac{1}{8} = \left( \frac{1}{2} \right)^{\frac{t}{5000}} \]

\[ \left( \frac{1}{2} \right)^3 = \left( \frac{1}{2} \right)^{\frac{t}{5000}} \]

\[ 3 = \frac{t}{1500} \]

\[ t = 4500 \]

\[ \begin{array}{c|c|c|c|c|}
\text{time} & 0 & 1500 & 3000 & 4500 \\
\hline
A(t) & 600 & 300 & 150 & 75 \\
\end{array} \]
2. Barium-122 has a half-life of 2 minutes. A fresh sample weighing 80 grams was obtained. If it takes 10 minutes to set up an experiment using barium-122, how much barium-122 will be left when the experiment begins?

\[ A(t) = A_0 \left( \frac{1}{2} \right)^{\frac{t}{2}} \]

\[ A(t) = 80 \left( \frac{1}{2} \right)^{\frac{10}{2}} \]

\[ A(t) = 80 \left( \frac{1}{2} \right)^5 \]

\[ = 80 \left( \frac{1}{32} \right) \]

\[ = 2.5 \text{ g} \]
3. When diving underwater, the light decreases as the depth of the diver increases. On a sunny day a diving team recorded 100% visibility at the surface but only 25% visibility 10 m below the surface. The team determined that the visibility of the dive could be modelled by the following half-life exponential function:

$$A(t) = A_o \left( \frac{1}{2} \right)^{\frac{t}{h}}$$

At what depth will the visibility be only half of what it was at the surface?

Find \( \text{half-life} = h \) ?

\[ A_0 = 100 \]
\[ A_t = 25 \]
\[ t = 10 \text{ m} \]

\[ A(t) = A_o \left( \frac{1}{2} \right)^{\frac{t}{h}} \]

\[ \frac{25}{100} = \left( \frac{1}{2} \right)^\frac{10}{h} \]

\[ \frac{1}{4} = \left( \frac{1}{2} \right)^\frac{10}{h} \rightarrow \text{make bases equal} \]

\[ \left( \frac{1}{2} \right)^2 = \left( \frac{1}{2} \right)^\frac{10}{h} \]

\[ \frac{2}{1} = \frac{10}{h} \]

\[ 2h = 10 \]

\[ h = 5 \text{ m} \]
4. The radioactive element cobalt-60 can be used to treat cancer patients. The percent of cobalt-60, \( A(t) \), left in a sample can be modeled by the half-life function

\[
A(t) = A_o \left( \frac{1}{2} \right)^{\frac{t}{5.3}}
\]

where \( t \) is the time, in years, after the initial time and \( A_o \) represent the initial amount, 100%, of the cobalt-60.

A) How long does it take for a sample of cobalt-60 to reduce to half its original amount?

5.3 years

B) What percent of cobalt-60 will remain in a sample after 10 years?

\[
A(t) = 100 \left( \frac{1}{2} \right)^{\frac{t}{5.3}}
\]

\[
A(t) = 100 \left( \frac{1}{2} \right)^{\frac{10}{5.3}}
\]

\[
A(t) = 27.1\%
\]

C) How long will it take until only 25% of the cobalt remains?

\[
\frac{25}{100} = \left( \frac{1}{2} \right)^{\frac{t}{5.3}}
\]

\[
\frac{1}{4} = \left( \frac{1}{2} \right)^{\frac{t}{5.3}}
\]

\[
\left( \frac{1}{2} \right)^2 = \left( \frac{1}{2} \right)^{\frac{t}{5.3}}
\]

\[
\frac{1}{4} = \frac{t}{5.3}
\]

\[
t = 10.6 \text{ years.}
\]
Doubling Time Problems (Growth)

If a quantity is doubling every so often...like in the E. Coli bacteria then we use a doubling time model for the exponential function.

\[ A(t) = A_o (2)^{t/h} \]

where...
- \( A_o \) represents the initial amount of the substance that was present
- \( t \) represents the time
- \( h \) represents the doubling time (amount of time taken for the substance to double)
- \( A(t) \) is the amount of the substance present at time \( t \)

5. The population of trout growing in a lake can be modeled by the function \( A(t) = A_o (2)^{t/h} \), where \( P(t) \) represents the number of trout and \( t \) represents the time in years after the initial count. How long will it take for there to be 6400 trout?

(A) What is the initial number of trout in the lake?

200

(B) How many years does it take for the trout to double?

5 years

(C) Determine how many years it will take for there to be 6400 trout.

Method 1: Algebraically

\[
\begin{align*}
A(t) &= 200 (2)^{t/5} \\
6400 &= 200 (2)^{t/5} \\
32 &= (2)^{t/5} \\
(2)^5 &= (2)^{t/5} \\
5 &= t/5 \\
t &= 25 \text{ years}
\end{align*}
\]

Method 2: Table of Values

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
\text{ } & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{P}(t) & 200 & 320 & 528 & 382 & 404 & 640 \\
\hline
\end{array}
\]

\[
\begin{align*}
\text{P}_{363} &= 364 \\
\text{P}_{382} &= 382 \\
\text{P}_{402} &= 404 \\
10-12, & 16, 15 \quad 7-8, & 12-13
\end{align*}
\]
6. A bacterial culture doubles in size every 12 hours. If there are 200 bacteria initially present,
   A) determine how many will be present 2 days from now.
   B) How long will it take for the culture to reach 51,200?

Given: \( A(t) = A_0 \left(2\right)^{t/h} \)

A) \( A_0 = 200 \)
   \( h = 12 \text{ hours} \)
   \( t = 2 \text{ days} \times 24 \text{h/day} \)
   \( = 48 \text{ h} \)

Formula:
\[
A(t) = 200 \left(2\right)^{t/12}
\]
\[
A(t) = 200 \left(2\right)^{48/12}
\]
\[
A(t) = 200 \left(2\right)^4
\]
\[
A(t) = 200 \left(16\right)
\]
\[
A(t) = 3200
\]

B) \( A(t) = 200 \left(2\right)^{t/12} \)
\[
\frac{51200}{200} = 200 \left(2\right)^{t/12}
\]
\[
256 = 2^{t/12}
\]
\[
8 = 2^{t/12}
\]
\[
8 = \frac{t}{12}
\]
\[
t = 96 \text{ h}
\]
Exponential Growth

\[ A(t) = P(1 + i)^n, \quad \text{— Given} \]

where

“\( A \)” is the future value

“\( P \)” is the principal (initial amount)

“\( i \)” is the interest rate per compounding period (expressed as a decimal) and

“\( n \)” is the number of compounding periods (i.e., the number of times interest is calculated in 1 year)

The above formula is sometimes written as \( A(t) = A_0(1 + i)^n \), if the interest rate is compounded annually.

Exponential Decay

\[ A(t) = A_0(1 - i)^n \quad \text{— Not Given} \]

where

\( A_0 \) is the initial amount

\( A(t) \) is the amount after time \( t \)

\( i \) is the annual interest rate expressed as a decimal
Growth

1. Shelly initially invests $500 and the value of the investment increases by 4% annually.

   A) Create a function to model this situation.

   \[ A(t) = P(1+i)^n \]

   \[ A(t) = 500 (1 + 0.04)^n \]

   \[ A(t) = 500 (1.04)^n \]

   B) How much money is in Shelly’s investment after 30 years?

   \[ A(t) = 500 (1.04)^{30} \]

   \[ A(t) = 1,621.70 \]

   C) How long does it take for Shelly’s money to double?

   \[ \frac{A(t)}{1000} = \frac{500 (1.04)^n}{500} \]

   \[ \frac{500}{500} = \frac{500 (1.04)^n}{500} \]

   \[ 2 = (1.04)^n \]

   \[ \text{Calculator} \quad y = 2 \quad y = 1.04^n \]

   \[ \text{Find intersection point. The x value is the soln.} \]

   \[ n = 17.7 \text{ years} \]
2. A Wayne Gretzky rookie card increases in value 2% every year. Recently, a card sold for $94,613 at auction. Determine the value of the card 12 years from now using an exponential model.

\[ A(t) = P(1 + i)^n \]

\[ A(t) = 94,613(1.02)^{12} = 119,992.16 \]
Decay

3. A car depreciates yearly by 20%. If the car was purchased for $24,500,

A) determine a model to represent the value of the car at any time \( t \).

\[
A(t) = P(1 - i)^n
\]

\[
A(t) = 24500(1 - 0.2)^n
\]

\[
A(t) = 24500(0.8)^n
\]

B) Use it to determine the value of the car at after 4 years?

\[
A(t) = 24500(0.8)^4
\]

\[
= \$10035.20
\]

C) When will the car be worth approximately $8000?

\[
\frac{8000}{24500} = \frac{24500(0.8)^n}{24500}
\]

\[
0.3265 = (0.8)^n
\]

\[ n = 5 \text{ years} \]
4. An adult takes 400 mg of ibuprofen. Each hour, the amount of ibuprofen in the person’s system decreases by about 29%. How much ibuprofen is left after 6 hours?

5. You drink a beverage with 120 mg of caffeine. Each hour, the caffeine in your system decreases by about 12%. How long until you have 10 mg of caffeine?